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Current Large Deviations for Asymmetric Exclusion Processes with Open Boundaries

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We study the large deviation functional of the current for the Weakly Asymmetric Simple Exclusion Process in contact with two reservoirs. We compare this functional in the large drift limit to the one of the Totally Asymmetric Simple Exclusion Process, in particular to the Jensen-Varadhan functional. Conjectures for generalizing the Jensen-Varadhan functional to open systems are also stated.

KEY WORDS: Non-equilibrium, particle systems, large deviations.
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1. INTRODUCTION

Recently, a theory has been developed to describe the current fluctuations of diffusive stochastic gas in their non-equilibrium steady state.[4,6,7,8,9,25,32] The large deviation functional of the current can be obtained in terms of a difficult variational problem; namely one needs to find the most likely time dependent density profile conditionally to the value of the current. So far some concrete informations have been derived: for example, explicit expressions of the current cumulants have been derived in Ref. (8) and dynamical phase transitions have been predicted in Refs. (6, 7, 9). In the present paper, we analyze this functional in the case of a one-dimensional diffusive lattice gas of length N, the simple exclusion process, with a weak asymmetry ν/N (this drift ν/N represents the effect of an external field acting on the particles from left to right). We will in particular consider the large drift limit $\nu \to \infty$ to investigate the relation between the hydrodynamic large

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deviation functional of the Weakly Asymmetric Simple Exclusion Process (WASEP) and the one of the Totally Asymmetric Simple Exclusion Process (TASEP) which was computed by Jensen-Varadhan.[24]

The density large deviation functional associated to the stationary measure of the TASEP in contact with reservoirs was computed in Refs. (16, 17). This computation was extended in Ref. (20) to the WASEP and the TASEP functional was recovered in the large drift limit (at least for some range of the parameters). This convergence was not a priori obvious as the hydrodynamic scalings in the WASEP and the TASEP are of different nature: for a system of size N, the time scales like N^2 in the WASEP and like N in the TASEP. Our goal here is to show that a similar convergence (from the WASEP to the TASEP) holds for the current large deviations for an open system in contact with two reservoirs (at least in some range of the values of the current). A similar convergence was already observed in Ref. (9) for the ring geometry.

In the present paper, we also discuss the crucial role played by the weak solutions of Burgers equation as in the Jensen-Varadhan theory.[24] It is well known that the weak solutions of Burgers equation are not unique.[36] A small viscosity enables to regularize the equation and to select the relevant weak solutions: from the PDE point of view, the others are disregarded as non-physical. These apparently non-physical weak solutions appear nevertheless as the optimal way to produce some current fluctuations and we show that the regularization by a small viscosity remains valid in the large deviation regime. We stress that the correspondence between the WASEP and the TASEP is obtained here at the macroscopic level (for the large deviation functionals). We are in fact not able to make a precise derivation starting from the microscopic models.

The paper is organized as follows. The WASEP in contact with reservoirs is introduced in Section 2. In Section 3, the current deviations of the WASEP are computed and in Section 5, they are compared to the Jensen-Varadhan theory which is outlined in Section 4. The generalization of the Jensen-Varadhan functional to open systems is also discussed in Section 5.

In the appendix we also calculate the probability of maintaining an anti-shock in the viscous Burgers equation, and in the formal limit $\nu = N$, we recover the Jensen-Varadhan functional.[24]

2. WEAKLY ASYMMETRIC SIMPLE EXCLUSION PROCESS

In this section, we briefly review some known results on the hydrodynamic behavior of the Weakly Asymmetric Simple Exclusion Process (WASEP).

2.1. The Microscopic Model

We consider the WASEP on the chain $\mathbb{D}_N = \{1, \ldots, N\}$ in contact with two reservoirs at the boundaries. A particle configuration is given by $\eta \in \{0, 1\}^N$, with $\eta_i = 1$ if a particle is at site i and $\eta_i = 0$ otherwise. The WASEP is a Markov process evolving according to the following dynamical rules. For a given $\nu > 0$ and N large enough, each particle attempts to jump to the right at rate $\frac{1}{2} + \frac{\nu}{2N}$ and to the left at rate $\frac{1}{2} - \frac{\nu}{2N}$. If the target site is occupied, the jump is disregarded.

Particles are injected or removed only at the reservoirs, i.e. at sites 1 and N. At the left boundary, i.e. at site 1, particles are created with rate c_1^+ and annihilated with rate c_1^-. Similarly at the right boundary, particles are created at site N with rate c_N^+ and annihilated with rate c_N^-.

Under the action of the dynamics, a particle current flows through the system. Let $Q(i, [s, s'])$ be the integrated current through the bond $(i, i+1)$ during the time interval $[s, s']$, i.e. the number of jumps from i to $i+1$ minus the number of jumps from $i+1$ to i. The space/time integrated current up to time s will be defined as

$$Q_s = \sum_{i=1}^{N-1} Q(i, [0, s]). \tag{2.1}$$

2.1.1. Hydrostatic

We first consider the stationary regime. The steady state density will be denoted by $\hat{\rho}$ and the steady state mean current by $\hat{q}$. At the macroscopic level, the domain $\mathbb{D}_N$ is mapped into $\mathbb{D} = [0, 1]$ and the local density of the steady state, at the macroscopic position x, can be defined as

$$\forall x \in]0, 1[, \qquad \hat{\rho}(x) = \lim_{\varepsilon \to 0} \lim_{N \to \infty} \left\langle \frac{1}{2\varepsilon N} \sum_{i=(x-\varepsilon)N}^{(x+\varepsilon)N} \eta_i \right\rangle_N, \tag{2.2}$$

where $\langle \cdot \rangle_N$ represents the expectation in the steady state. At the boundaries, $\hat{\rho}$ is given by the rates associated to the reservoirs

$$\hat{\rho}(0) = \rho_a = \frac{c_1^+}{c_1^- + c_1^+} \quad \text{and} \quad \hat{\rho}(1) = \rho_b = \frac{c_N^+}{c_N^- + c_N^+}. \tag{2.3}$$

The local density satisfies Fick's law

$$\forall x \in]0, 1[, \qquad \frac{1}{2}\Delta_x \hat{\rho} - \nu \nabla_x(\sigma(\hat{\rho})) = 0\,, \tag{2.4}$$

with

$$\sigma(\rho) = \rho(1 - \rho). \tag{2.5}$$

The hydrostatic Eq. (2.4) has been derived for a broad class of stochastic models[3,21,22,26,30] and in particular for the WASEP in Ref. (20).

The mean current can also be deduced from (2.4) (see Ref. (21))

$$\hat{q} = \lim_{N\to\infty} \lim_{s\to\infty} \left\langle \frac{Q_s}{s} \right\rangle_N = -\frac{1}{2}\nabla_x \hat{\rho} + \nu\sigma(\hat{\rho})\,. \tag{2.6}$$

As the previous expectation is taken wrt the steady state, the formula above is valid at any time s. According to Fick's law the current through each bond is of the order $1/N$, where N is the length of the system. The previous scaling comes from the fact that Q_s is defined as the sum of the contributions of the N bonds (2.1).

The mean current can be explicitly computed from (2.6). As an example, let us examine the case $\rho_a > \frac{1}{2} > \rho_b$ which becomes the *maximum current phase* in the TASEP, that is in the limit of a strong asymmetry.[14,31,34,35] Equation (2.6) implies that

$$2 = \int_{\rho_b}^{\rho_a} \mathrm{d}\rho \, \frac{1}{\hat{q} - \nu\sigma(\rho)} = \int_{v_b}^{v_a} \mathrm{d}v \, \frac{1}{\hat{q} - \frac{\nu}{4} + \nu v^2},$$

where we used the change of variables $\rho \to 1/2 + v$ with $v_a = \rho_a - 1/2,\ v_b = \rho_b - 1/2$. This leads to the following asymptotic for large ν

$$\hat{q} = \frac{\nu}{4} + \frac{\pi^2}{4\nu} + o\left(\frac{1}{\nu}\right). \tag{2.7}$$

2.1.2. Hydrodynamic

We consider now the evolution of the stochastic process and denote by $\eta(s)$ the configuration at time s. Let φ be a macroscopic density profile. The evolution of the particle system over the time interval $[0, s]$ and starting with initial data close to φ has the probability $\mathbb{P}^{\varphi}_{N,s}$ (which depends on ν as well). More precisely, $\mathbb{P}^{\varphi}_{N,s}$ is such that if the initial data are randomly chosen wrt the Bernoulli product measure with mean $\varphi(i/N)$ at site i, then the time evolution is given by the Markov dynamics. To study the hydrodynamic limit, we introduce the diffusive scaling

$$x = \frac{i}{N}, \qquad t = \frac{s}{N^2}. \tag{2.8}$$

At the macroscopic level, the particle system is identified to the macroscopic density profile ρ defined for any $(x, t) \in \mathbb{D} \times [0, T]$ by

$$\rho(x, t) = \lim_{\varepsilon\to 0} \lim_{N\to\infty} \frac{1}{2\varepsilon N} \sum_{i=(x-\varepsilon)N}^{(x+\varepsilon)N} \eta_i(N^2 t). \tag{2.9}$$

If initially the discrete system is close (after rescaling) to the macroscopic profile $\rho(x, 0) = \varphi(x)$ then for any $t > 0$ the rescaled system remains close to the density profile solution of

$$\forall (x,t) \in \mathbb{D} \times [0,T], \qquad \partial_t \rho = \frac{1}{2}\Delta\rho - \nu \nabla_x(\sigma(\rho)) \quad \text{with}$$
$$\rho(0,t) = \rho_a, \ \rho(1,t) = \rho_b, \tag{2.10}$$

where φ is the initial data and $\sigma(\rho) = \rho(1-\rho)$. This holds with a probability $\mathbb{P}^{\varphi}_{N,TN^2}$ close to 1.[3,21,28,38]

For any $(x,t) \in \mathbb{D} \times [0,T]$, a local version of the space-time current can be defined as a space-time average on the microscopic boxes $\{(x-\varepsilon)N, (x+\varepsilon)N\} \times [tN^2, (t+\varepsilon)N^2]$

$$j(x,t) = \lim_{\varepsilon \to 0} \lim_{N \to \infty} \frac{1}{2\varepsilon^2 N^2} \sum_{i=(x-\varepsilon)N}^{(x+\varepsilon)N} Q(i, [tN^2, (t+\varepsilon)N^2]). \tag{2.11}$$

The scaling in (2.11) can be understood as follows: j is the rescaled current while the microscopic current through each bond is j/N. Then summing over $2\varepsilon N$ bonds during a time interval εN^2 implies that one has to normalize by a factor $2\varepsilon^2 N^2$. The conservation of the number of particles at the microscopic level becomes through (2.9) and (2.11)

$$\partial_t \rho = -\nabla_x j.$$

2.2. Large Deviations

The hydrodynamic Eq. (2.10) describes the typical behavior under the stochastic evolution. The probability of observing a different trajectory can also be computed. In fact, one can even estimate the probability of observing a given current of particles $j(x,t)$ and its corresponding density profile defined by

$$\forall (x,t) \in \mathbb{D} \times [0,T], \qquad \partial_t \rho = -\nabla_x j, \tag{2.12}$$

with some initial data $\rho(x,0) = \varphi(x)$ in $\mathbb{D}$ (Note that the current cannot be completely arbitrary: it should be such that the density (2.12) remains in [0, 1] and is equal to ρ_a and ρ_b at the boundaries at any time $t > 0$.)

The probability of the event $\{(j,\rho)\}$ that the rescaled particle system has a current j and remains close to the density profile ρ introduced in (2.12) is given asymptotically in N by

$$\mathbb{P}^{\varphi}_{N,TN^2}(j,\rho) \approx \exp\left(-N\mathcal{I}^{\nu}_{[0,T]}(j,\rho) + o(N)\right), \tag{2.13}$$

and the large deviation functional is

$$\mathcal{I}^{\nu}_{[0,T]}(j,\rho) = \int_0^T \mathrm{d}t \int_{\mathbb{D}} \mathrm{d}x \, \frac{\left(j(x,t) - \nu\sigma(\rho(x,t)) + \frac{1}{2}\nabla_x \rho(x,t)\right)^2}{2\,\sigma(\rho(x,t))}. \tag{2.14}$$

This can be interpreted as a local Gaussian fluctuation of the current with a variance $\sigma(\rho(x,t))$ depending on the local density.

The hydrodynamic large deviation theory was originally introduced in Refs. (27, 28, 38) to estimate the probability of events corresponding to atypical space/time density profiles. Recent developments[6,7,8,25,32] led to the more general expression (2.14) which enables one to control a deviation of the density as well as the associated current (see the appendix of Ref. (9) for a heuristic derivation). Note that the current $j(x,t)$ appearing in (2.13–2.14) is an average over a long microscopic time of order N^2 (see (2.11)). Recent works[18,19] have also considered the correlations between the density and the instantaneous current. At present we do not see any direct connection between their distribution of the instantaneous current and our $j(x,t)$.

3. CURRENT LARGE DEVIATIONS FOR A SYSTEM WITH RESERVOIRS

In this Section, we investigate the current large deviations for the WASEP in contact with 2 reservoirs at densities ρ_a and ρ_b.

3.1. General Estimates

We are interested in the asymptotic probability of observing a deviation of the integrated current over a very long time interval $[0, T N^2]$, i.e. in the event that

$$\frac{Q_{T N^2}}{T N^2} = \frac{1}{T}\int_0^T \mathrm{d}t \int_0^1 \mathrm{d}x \; j(x,t) = \mathcal{J} \tag{3.1}$$

for a very large macroscopic time T (the correspondence between Q and j was stated in (2.11)). According to the large deviation principle (2.13), one has to minimize the functional (2.14) over the time dependent density and current profiles under the mean current constraint (3.1).[6,7,9] Suppose that the optimal way to achieve this deviation is by maintaining the density profile close to some optimal time independent density profile. Then, the variational problem simplifies and one has for T large

$$\mathbb{P}^{\varphi}_{N,TN^2}\left(\frac{Q_{T N^2}}{T N^2} \sim \mathcal{J}\right) \approx \exp(-NTG(\mathcal{J})),$$

with

$$G(\mathcal{J}) = \inf_{\rho}\left\{\int_0^1 dx \; \frac{\left(\mathcal{J} - \nu\sigma(\rho(x)) + \frac{1}{2}\rho'(x)\right)^2}{2\,\sigma(\rho(x))}\right\}, \tag{3.2}$$

where the infimum is taken over all the density profiles $\rho(x)$ with boundary conditions ρ_a, ρ_b prescribed by the reservoirs. For simplicity, we wrote $\rho' = \nabla_x \rho$.

This stationary regime assumption was already made in Ref. (8) when $\nu = 0$ and led to an explicit prediction for all the cumulants of the integrated current Q_T in diffusive lattice gases. It was however understood in Ref. (6, 7) that the stationarity assumption may not be satisfied for some diffusive lattice gases. In this case, a space/time dependent current is more favorable than a constant current and $G(\mathcal{J})$ does not give the correct order for observing the mean current $\mathcal{J} = \frac{1}{T}\int_{t=0}^{T}\int_{\mathbb{D}} j(x,t)$ (see Refs. (6, 7, 9)).

Given an arbitrary density profile φ at time $t = 0$, the expression (3.2) can nevertheless be interpreted as the asymptotic cost for observing a constant current $j(x,t) = \mathcal{J}$ for *all* times t in $[t_0, T]$ with $t_0 \ll T$. The evolution during the time interval $[0, t_0]$ does not contribute to (3.2), but it allows the system to reach the optimal density profile $\rho(x)$ in (3.2) whatever the initial profile φ is.

Let us first show that the optimal profile ρ of the variational problem (3.2) satisfies the relation

$$\left(\frac{1}{2}\rho'\right)^2 = (\mathcal{J} - \nu\sigma(\rho))^2 + 2K\sigma(\rho), \tag{3.3}$$

where K is an integration constant determined by the boundary conditions ρ_a, ρ_b (this formula is an extension of Eq. (15) obtained in Ref. (8)). To see this, we write for a given density profile ρ

$$\int_0^1 \mathrm{d}x \,\frac{\left(\mathcal{J} - \nu\sigma(\rho) + \frac{1}{2}\rho'\right)^2}{2\,\sigma(\rho)} = \frac{1}{2}\int_0^1 \mathrm{d}x\,\left(X(\rho) + \frac{(\rho')^2}{4\sigma(\rho)}\right) - \int_{\rho_b}^{\rho_a} \mathrm{d}u\,\frac{\mathcal{J} - \nu\sigma(u)}{2\sigma(u)}, \tag{3.4}$$

where $X(\rho) = \frac{(\mathcal{J}-\nu\sigma(\rho))^2}{\sigma(\rho)}$. The corresponding Euler equation is

$$X'(\rho) - \frac{\rho''}{2\sigma(\rho)} + \frac{(\rho')^2\sigma'(\rho)}{4\sigma(\rho)^2} = 0.$$

Multiplying by ρ' we obtain

$$\nabla X(\rho) - \nabla\left(\frac{(\rho')^2}{4\sigma(\rho)}\right) = 0.$$

This completes the derivation of (3.3).

In order to obtain more precise results, we are going now to specify the current deviation $\mathcal{J}$ as well as the boundary conditions.

3.2. Maximal Current Phase

As an example, let us consider the case $\rho_a > 1/2 > \rho_b$, which becomes in the large ν limit the "maximal current phase." According to (2.7), the mean current is of order $\nu/4$. We are interested in the large ν asymptotic of the large deviation functional $G(\nu q)$ (given in (3.2)) for current deviations of the form

$$\mathcal{J} = \nu q.$$

3.2.1. Case 1 (Current Reduction): q < 1/4

Fix $\alpha > 1/2$. A constant profile at density α has a typical current $q = \sigma(\alpha) = \sigma(1-\alpha) < 1/4$. As we will see the optimal profile to achieve a current deviation q concentrates close to either α or $1-\alpha$ as ν tends to infinity.

Setting $K = \nu^2 k$, the Eq. (3.3) of the minimizers of the functional $G(\nu q)$ can be rewritten

$$\left(\frac{1}{2\nu}\rho'\right)^2 = (q - \sigma(\rho))^2 + 2k\sigma(\rho). \tag{3.5}$$

Different behaviors occur depending on the value of q, i.e. on the location of α wrt the reservoir densities ρ_a and ρ_b.

- *Suppose that* $\alpha \in [\rho_b, \rho_a]$ *or* $1-\alpha \in [\rho_b, \rho_a]$: As the optimal density profile goes smoothly from ρ_a to ρ_b, it has to take the value α (or $1-\alpha$) and (3.5) implies that $k \geq 0$. This implies that the optimal profile is decreasing. Integrating (3.5) leads to

$$2\nu = \int_{\rho_b}^{\rho_a} \mathrm{d}\rho \frac{1}{\sqrt{(q-\sigma(\rho))^2 + 2k\sigma(\rho)}}. \tag{3.6}$$

  Thus as ν tends to infinity, k must vanish (in fact from (3.6), one can show that k vanishes exponentially fast wrt ν). The optimal profile remains essentially equal to α or $1-\alpha$.
  The expression (3.5) of the optimal profile combined with (3.2) leads to

$$G(\nu q) = \nu \int_{\rho_b}^{\rho_a} \mathrm{d}\rho \frac{1}{2\sigma(\rho)} \left[\frac{(q-\sigma(\rho))^2 + k\sigma(\rho)}{\sqrt{(q-\sigma(\rho))^2 + 2k\sigma(\rho)}} - (q - \sigma(\rho))\right]. \tag{3.7}$$

  As $k \to 0$ when $\nu \to \infty$

$$\lim_{k\to 0} \frac{(q-\sigma(\rho))^2 + k\sigma(\rho)}{\sqrt{(q-\sigma(\rho))^2 + 2k\sigma(\rho)}} = \begin{cases} \sigma(\rho) - q, & \text{if } \rho \in [1-\alpha, \alpha], \\ q - \sigma(\rho), & \text{otherwise.} \end{cases} \tag{3.8}$$

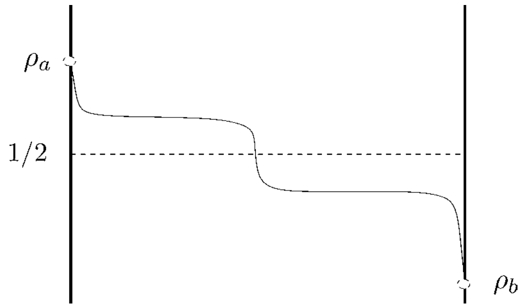

Fig. 1. Optimal profile associated to the current deviation (3.10) for large ν.

Let us define $\rho^+ = \min\{\alpha, \rho_a\}$ and $\rho^- = \max\{1-\alpha, \rho_b\}$, then only the densities in $[\rho^-, \rho^+]$ contribute to the asymptotic

$$\lim_{\nu\to\infty} \frac{1}{\nu} G(\nu q) = \int_{\rho^-}^{\rho^+} \mathrm{d}\rho \, \frac{(\sigma(\rho) - q)}{\sigma(\rho)}$$
$$= (\rho^+ - \rho^-) - q \left[\log \frac{\rho^+}{1-\rho^+} - \log \frac{\rho^-}{1-\rho^-} \right]. \tag{3.9}$$

where we used that $\sigma(\rho) = \rho(1-\rho)$.

In particular, for $[1-\alpha, \alpha] \subset [\rho_b, \rho_a]$, this leads to the first main result of this paper

$$\lim_{\nu\to\infty} \frac{1}{\nu} G(\nu q) = (g(1-\alpha) - g(\alpha)), \tag{3.10}$$

where

$$g(u) = u(1-u) \log \frac{u}{1-u} - u. \tag{3.11}$$

In this case the optimal density profile is essentially a piecewise constant function with three jumps over regions of width $1/\nu$: a boundary layer near the left reservoir where the density varies from ρ_a to α, another one from $1-\alpha$ to ρ_b near the right reservoir and a jump from α to $1-\alpha$ (see Fig. 1). The sharp discontinuity between α and $1-\alpha$ will be interpreted in Section 5.1 as a convergence to an anti-shock and the asymptotic cost (3.10) coincides with the Jensen-Varadhan functional which will be defined in (4.8).

- *Suppose that* $[\rho_b, \rho_a] \subset [1-\alpha, \alpha]$: The optimal profile has to be close to α or $1-\alpha$ otherwise $G(\nu q)$ would scale like ν^2 (see (3.4)), thus the optimal profile cannot be always decreasing. We first assume that it initially increases from ρ_a to a local maximum γ_ν depending on ν. According to

(3.5), one has

$$0 = (q - \sigma(\gamma_\nu))^2 + 2k\sigma(\gamma_\nu). \tag{3.12}$$

The density cannot increase beyond γ_ν otherwise the RHS of (3.5) would be negative. Thus the profile $\rho(x)$ increases from ρ_a to γ_ν in the interval $[0, x_\nu]$ and then decreases to ρ_b in the interval $[x_\nu, 1]$. The analogous of (3.6) can be rewritten as

$$2\nu\, x_\nu = \int_{\rho_a}^{\gamma_\nu} \mathrm{d}\rho \frac{1}{\sqrt{(q-\sigma(\rho))^2 + 2k\sigma(\rho)}} \quad \text{and}$$

$$2\nu\, (1 - x_\nu) = \int_{\rho_b}^{\gamma_\nu} \mathrm{d}\rho \frac{1}{\sqrt{(q-\sigma(\rho))^2 + 2k\sigma(\rho)}}.$$

For large ν, the important feature is the singularity at γ_ν which implies that γ_ν has to converge to α and k to 0 as ν diverges. In this case, the profile concentrates close to α. Then (3.4) becomes

$$\nu \int_{\rho_a}^{\gamma_\nu} \mathrm{d}\rho \frac{1}{2\sigma(\rho)} \left[\frac{(q-\sigma(\rho))^2 + k\sigma(\rho)}{\sqrt{(q-\sigma(\rho))^2 + 2k\sigma(\rho)}} + (q - \sigma(\rho)) \right]$$
$$+ \nu \int_{\rho_b}^{\gamma_\nu} \mathrm{d}\rho \frac{1}{2\sigma(\rho)} \left[\frac{(q-\sigma(\rho))^2 + k\sigma(\rho)}{\sqrt{(q-\sigma(\rho))^2 + 2k\sigma(\rho)}} - (q - \sigma(\rho)) \right].$$

In the limit $\nu \to \infty$, a computation similar to (3.9) applies: the contribution of the reservoir ρ_a vanishes and the asymptotic cost is

$$\int_{\rho_b}^{\alpha} \mathrm{d}\rho \; \frac{(\sigma(\rho) - q)}{\sigma(\rho)} = \alpha - \rho_b + \alpha(1-\alpha) \left(\log \frac{\rho_b}{(1-\rho_b)} - \log \frac{\alpha}{(1-\alpha)} \right). \tag{3.13}$$

Similarly, if the profile first decreases from ρ_a to $1 - \alpha$ and then increases to ρ_b, then the asymptotic cost is

$$\int_{1-\alpha}^{\rho_a} \mathrm{d}\rho \; \frac{(\sigma(\rho) - q)}{\sigma(\rho)} = \rho_a - (1-\alpha) - \alpha(1-\alpha) \times \left(\log \frac{\rho_a}{(1-\rho_a)} - \log \frac{(1-\alpha)}{\alpha} \right). \tag{3.14}$$

Therefore, we can state our second main result. If one defines the reservoir contribution by

$$\forall u, v \in [0, 1],$$
$$F^{\text{res}}(u, v) = u - v - v(1-v) \left(\log \frac{u}{(1-u)} - \log \frac{v}{(1-v)} \right), \tag{3.15}$$

then the optimal cost of the current deviation is given by

$$\lim_{\nu\to\infty} \frac{1}{\nu} G(\nu q) = \min\left\{F^{\rm res}(\rho_a, 1-\alpha), -F^{\rm res}(\rho_b, \alpha)\right\}. \qquad (3.16)$$

where as before $q = \sigma(\alpha) = \alpha(1-\alpha)$. We remark that the cost is due to the boundary effects and that the contributions from the left and the right reservoirs decouple: the first term in (3.16) is a cost for a boundary layer near the left reservoir with a density jump from ρ_a to $1-\alpha$ whereas the second term is the cost for a boundary layer near the right reservoir with a density jump from α to ρ_b. One can trigger a phase transition between the optimal profile close to α and the one close to $1-\alpha$ by varying ρ_a and ρ_b (the critical curve is $\rho_a = 1-\rho_b$). This provides an additional example of a non-equilibrium phase transition associated to a current large deviation.[6,7,9]

3.2.2. Case 2 (Current Increase): $q > 1/4$

Once again the optimal profile is determined by (3.5). The optimal profile decreases from ρ_a to ρ_b and Eq. (3.6) applies. The minimum of the derivative ρ' (see (3.5)) is reached at $\rho = 1/2$, so that (3.6) implies that for large ν

$$k = -2\left(q - \frac{1}{4}\right)^2 + \varepsilon_\nu, \qquad (3.17)$$

where ε_ν vanishes as ν tends to infinity. The functional (3.7) can be rewritten, using (3.6)

$$G(\nu q) = -\nu^2 k + \nu \int_{\rho_b}^{\rho_a} {\rm d}\rho \frac{1}{2\sigma(\rho)} \left[\sqrt{(q-\sigma(\rho))^2 + 2k\sigma(\rho)} - (q - \sigma(\rho))\right].$$

This leads to

$$G(\nu q) \sim 2\nu^2 \left(q - \frac{1}{4}\right)^2 \quad \text{and} \quad \lim_{\nu\to\infty} \frac{1}{\nu} G(\nu q) = \infty. \qquad (3.18)$$

A deviation $q > 1/4$ leads to asymptotic of $G(\nu q)$ in ν^2 in contrast to a deviation $q \in [0, 1/4]$ where $G(\nu q)$ scales like ν.

3.3. Boundary Effects

The current deviations select very specific density profiles. We are going now to estimate the large deviations associated to atypical density profiles instead of atypical currents. The main motivation is to compute the boundary contribution for a large drift ν.

We first fix $\rho_a > 1/2$ and let $\rho_b = \gamma$. We are interested in evaluating the probability of maintaining a density profile close to γ over a very long time T, i.e.

$$\mathbb{P}^{\gamma}_{N,TN^2}\left(\forall t \in [0,T], \quad \int_0^1 \mathrm{d}x\, (\rho(x,t)-\gamma)^2 \le \varepsilon\right) \approx \exp\left(-TN\mathcal{G}^L(\nu,\rho_a,\gamma)\right), \tag{3.19}$$

where the initial data is the profile uniformly equal to γ. The choice $\rho_b = \gamma$ is made in order to suppress the boundary effect of the right reservoir so that the main contribution will depend only on the left reservoir at density ρ_a. The functional $\mathcal{G}^L$ should be interpreted as the contribution of the left reservoir.

The optimal way to achieve the deviation (3.19) could involve a time dependent solution $\rho(x,t)$. We are not able to exclude this eventuality, but we make the assumption that this is not the case. The large deviations (2.13) restricted to time independent density profiles imply that

$$\mathcal{G}^L(\nu,\rho_a,\gamma) = \inf_{\mathcal{J},\rho}\left\{\int_0^1 \mathrm{d}x\, \frac{\left(\mathcal{J} - \nu\sigma(\rho(x)) + \frac{1}{2}\rho'(x)\right)^2}{2\,\sigma(\rho(x))}\right\}, \tag{3.20}$$

where the infimum is taken over the constant currents $\mathcal{J}$ and the density profiles ρ such that $\int_0^1 \mathrm{d}x\,(\rho(x)-\gamma)^2 \le \varepsilon$.

We are going to show that

$$\forall \rho_a > \frac{1}{2}, \lim_{\varepsilon\to 0}\lim_{\nu\to\infty}\frac{1}{\nu}\mathcal{G}^L(\nu,\rho_a,\gamma) = \begin{cases} F^{\mathrm{res}}(\rho_a,\gamma), & \text{if } \gamma \in [0, 1-\rho_{\mathrm{a}}] \\ g(\gamma)-g(1-\gamma), & \text{if } \gamma \in [1-\rho_{\mathrm{a}}, 1/2] \\ 0, & \text{if } \gamma \in [1/2, 1] \end{cases} \tag{3.21}$$

where F^{res} was introduced in (3.15) and g in (3.10). This can be intuitively understood in view of the previous results on the current large deviations. $F^{\mathrm{res}}(\rho_a,\gamma)$ was computed in (3.15) as the optimal cost for producing a current $\nu\sigma(\gamma)$ with an optimal profile close to $\gamma < 1-\rho_a$, so that the constraint $\{\int_0^1 \mathrm{d}x\,(\rho(x)-\gamma)^2 \le \varepsilon\}$ does not change the asymptotic (3.16). This implies that for $\gamma \in [0, 1-\rho_a]$, the boundary cost is also given by $F^{\mathrm{res}}(\rho_a,\gamma)$. If $\gamma \in [1-\rho_a, \rho_a]$ then the optimal profile to achieve a current $\nu\sigma(\gamma)$ is concentrated close to $1-\gamma$ and γ (see (3.10)). In this case, the cost is essentially given by the sharp jump between $1-\gamma$ and γ (see Fig. 1). To satisfy the density constraint, it is enough to shift the jump close to the reservoir ρ_a so that all the energy becomes concentrated in the boundary layer (see Fig. 2). Shifting the discontinuity has a cost much smaller than ν thus this cost disappears in the large ν asymptotic and we recover only the bulk contribution $g(\gamma)-g(1-\gamma)$. Finally for $\gamma \in [1/2, 1]$, the stationary profile is equal to γ so that there is no cost to maintain the density γ.

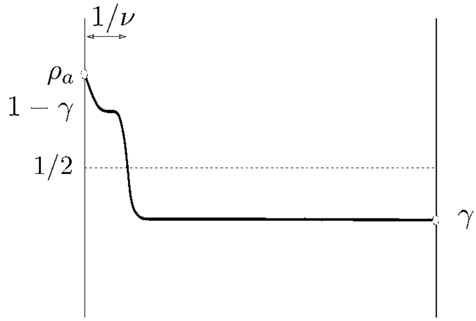

Fig. 2. The boundary layer is made of two parts which are at a distance of order $1/\nu$ of the left reservoir.

We turn now to the derivation of (3.21). The optimal current $\mathcal{J}$ (in the variational problem (3.20)) is related to the optimal profile ρ by

$$\mathcal{J}\int_0^1 \frac{dx}{\sigma(\rho(x))} = \int_0^1 \mathrm{d}x\, \frac{\nu\sigma(\rho(x)) - \frac{1}{2}\rho'(x)}{\sigma(\rho(x))} = \nu + \int_\gamma^{\rho_a} \frac{d\rho}{2\,\sigma(\rho)}. \quad (3.22)$$

The density constraint implies that for ν large, the current is of the order $\mathcal{J} = \nu q$, where $q \in I_\delta = [\sigma(\gamma) - \delta, \sigma(\gamma) + \delta]$ and δ vanishes as ε tends to 0. Therefore, one can substitute the density constraint by a current constraint and derive a lower bound for $\mathcal{G}^L(\nu, \rho_a, \gamma)$

$$\mathcal{G}^L(\nu, \rho_a, \gamma) \geq \inf_{q\in I_\delta} \inf_{\rho} \left\{ \int_0^1 \mathrm{d}x\, \frac{\left(\nu q - \nu\sigma(\rho(x)) + \frac{1}{2}\rho'(x)\right)^2}{2\,\sigma(\rho(x))} \right\} \geq \inf_{q\in I_\delta} G(\nu q),$$

where G was introduced in (3.2). For $\gamma \in [0, 1/2]$, then the asymptotic of $G(\nu q)$ follow from (3.10), (3.14). When δ tends to 0, we deduce by continuity that the RHS of (3.21) is a lower bound for the asymptotic of $\mathcal{G}^L(\nu, \rho_a, \gamma)$. For $\gamma \in [1/2, 1]$, a cancellation similar to the one in (3.9) occurs and we recover 0 as a lower bound.

In order to derive an upper bound for $\mathcal{G}^L(\nu, \rho_a, \gamma)$, it is enough to compute the asymptotic cost of well chosen density profiles. Let us simply consider the case $\gamma \in [1 - \rho_a, \rho_a]$ and define the density profile ψ

$$\left(\frac{1}{2\nu}\psi'\right)^2 = (\sigma(\gamma) - \sigma(\psi))^2 + 2k\sigma(\psi) + \varepsilon'(\psi - \gamma)^2, \quad (3.23)$$

where k is chosen such that the boundary conditions are satisfied. We remark that ψ is essentially given by Eq. (3.5) of the optimal profile associated to a current $\sigma(\gamma)$. On the other hand the profile given by (3.5) is not necessarily concentrated close to the density γ as there might be a jump from $1 - \gamma$ to γ (see Fig. 1). The parameter ε' ensures that the discontinuity between $1 - \gamma$ and γ is located close to the left boundary (see Fig. 2). By tuning ε' appropriately wrt ε, we see that the profile ψ concentrates to the density γ for large ν, so that the density constraint is satisfied

$$\int_0^1 dx \, \frac{\left(\nu q - \nu\sigma(\psi(x)) + \frac{1}{2}\psi'(x)\right)^2}{2\,\sigma(\psi(x))} \geq \mathcal{G}^L(\nu, \rho_a, \gamma).$$

Proceeding as in (3.7), (3.9), we deduce that the asymptotic cost of the profile (3.23) converges to $g(\gamma) - g(1-\gamma)$. Similar arguments for the other values of γ enables to complete the proof of (3.21).

If $\rho_a < 1/2$, then similar computations lead to

$$\lim_{\varepsilon\to 0}\lim_{\nu\to\infty} \frac{1}{\nu}\mathcal{G}^L(\nu, \rho_a, \gamma) = \begin{cases} F^{\text{res}}(\rho_a, \gamma), & \text{if } \gamma \in [0, 1/2] \\ F^{\text{res}}(\rho_a, 1-\gamma), & \text{if } \gamma \in [1/2, 1-\rho_{\text{a}}] \\ 0, & \text{if } \gamma \in [1-\rho_{\text{a}}, 1] \end{cases}, \quad (3.24)$$

where F^{res} was introduced in (3.15).

The boundary cost of the right reservoir can be deduced by symmetries (particle-hole,left-right). For $\rho_b > 1/2$, the asymptotic cost of a discontinuity γ at the right reservoir is given by

$$\lim_{\varepsilon\to 0}\lim_{\nu\to\infty} \frac{1}{\nu}\mathcal{G}^R(\nu, \gamma, \rho_b) = \begin{cases} -F^{\text{res}}(\rho_b, \gamma), & \text{if } \gamma \in [1/2, 1] \\ -F^{\text{res}}(\rho_b, 1-\gamma), & \text{if } \gamma \in [1-\rho_{\text{b}}, 1/2]. \\ 0, & \text{if } \gamma \in [0, 1-\rho_{\text{b}}] \end{cases} \quad (3.25)$$

For $\rho_b < 1/2$

$$\lim_{\varepsilon\to 0}\lim_{\nu\to\infty} \frac{1}{\nu}\mathcal{G}^R(\nu, \gamma, \rho_b) = \begin{cases} -F^{\text{res}}(\rho_b, \gamma), & \text{if } \gamma \in [1-\rho_{\text{b}}, 1] \\ g(1-\gamma) - g(\gamma), & \text{if } \gamma \in [1/2, 1-\rho_{\text{b}}]. \\ 0, & \text{if } \gamma \in [0, 1/2] \end{cases} \quad (3.26)$$

In Section 3.2, we focused on the current large deviation for $\rho_a > 1/2 > \rho_b$. For other choices of the reservoir densities, similar computation would lead to asymptotic costs which can be expressed also in terms of (3.21), (3.24–3.26). Further interpretation of the boundary effects is postponed until Section 5.2.

4. JENSEN-VARADHAN FUNCTIONAL

In this Section, we recall some properties of the TASEP and discuss the hydrodynamic large deviations which were derived by Jensen and Varadhan in Ref. (24) The connection between this functional and the previous results obtained for the WASEP will be detailed in Section 5.

4.1. TASEP

Following the notation introduced for the WASEP, we consider the TASEP on the domain $\mathbb{D}_N = \{1, \ldots, N\}$ in contact with two reservoirs at the boundaries and denote a particle configuration at time s by $\eta(s) \in \{0, 1\}^N$. For the TASEP, the particles only jump to the right with a constant rate 1 if the neighboring site on their right is empty. The reservoirs follow the same dynamical rules as the ones introduced in Section 2.1. Let $\mathbb{Q}^{\varphi}_{N,s}$ be the probability measure associated to the TASEP on $\mathbb{D}_N$ during the microscopic time interval $[0, s]$ and starting from the Bernoulli product measure with local density at site i equal to $\varphi(i/N)$.

The hydrodynamic behavior obeys a different scaling from the one of the WASEP

$$x = \frac{i}{N}, \qquad t = \frac{s}{N}. \tag{4.1}$$

At the macroscopic level, the particle system is identified to the macroscopic density profile

$$\rho(x, t) = \lim_{\varepsilon \to 0} \lim_{N \to \infty} \frac{1}{2\varepsilon N} \sum_{i=(x-\varepsilon)N}^{(x+\varepsilon)N} \eta_i(Nt).$$

If initially the discrete system is close (after rescaling) to the macroscopic profile $\rho(x, 0)$ then for any $t > 0$, it was proven in Ref. (1) (see also Refs. (28, 33)), that the rescaled system remains close to the density profile evolving according to Burgers equation

$$\forall (x, t) \in \mathbb{D} \times [0, T], \quad \partial_t \rho + \nabla_x(\sigma(\rho)) = 0 \quad \text{with} \quad \sigma(\rho) = \rho(1 - \rho). \tag{4.2}$$

For general initial conditions, the solution of this PDE is not well defined as shocks may occur after a finite time in the bulk and at the boundaries. For simplicity, we will not discuss the boundary effects and focus on the bulk singularities. A detailed mathematical study of the PDE (4.2) can be found in Refs. (2, 36, 37). Notice also that the hydrodynamic limit can be derived beyond the shocks for a broad class of lattice gases.[1,33]

In order to take into account the shocks in the bulk, Burgers equation has to be rephrased in terms of weak solutions. Let Φ be a smooth function with compact support strictly included in $\mathbb{D} \times [0, T]$, i.e. there is $\delta > 0$ such that

$$\forall x \notin [\delta, 1-\delta], \quad \Phi(x, t) = 0 \quad \text{and} \quad \forall x \in \mathbb{D}, \quad \Phi(x, 0) = \Phi(x, T) = 0.$$

Integrating (4.2) by parts leads to

$$\int_0^T \mathrm{d}t \int_0^1 \mathrm{d}x \, \{\rho \, \partial_t \Phi + \sigma(\rho) \, \nabla_x \Phi\} = 0. \tag{4.3}$$

A trajectory ρ is said to be a *weak solution* of (4.2) if it satisfies (4.3) for any test function Φ with compact support strictly included in $\mathbb{D} \times [0, T]$. This formulation is not sufficient to prescribe the behavior at the boundary, nevertheless we will be mainly interested in bulk properties for which it is enough to consider (4.3).

For a given initial data, there is no uniqueness of the weak solutions. Let us consider, as an example, a weak solution on $\mathbb{R}$. We define a density profile with a discontinuity a, b moving at speed v, i.e.

$$\rho(x,t) = \begin{cases} a, & \text{if } x < vt, \\ b, & \text{if } x > vt. \end{cases} \tag{4.4}$$

This profile is a solution of (4.3) for any a, b in [0, 1] provided that the velocity is given by

$$v = \frac{\sigma(a) - \sigma(b)}{a - b}. \tag{4.5}$$

If $a < b$, the previous density profile (4.4) is a physical solution. For $a > b$, the initial data with the discontinuity a, b at 0 evolves as a rarefaction fan so that the previous density profile should be disregarded and considered as a non physical event. In the following, we will refer to the discontinuity $a < b$ as a *shock* and $a > b$ as an *anti-shock*.[36]

A way to select the physical solutions is to add a small viscosity ε and to recover Burgers equation in the limit $\varepsilon \to 0$

$$\forall (x,t) \in \mathbb{D} \times [0, T], \qquad \partial_t \rho + \nabla_x(\sigma(\rho)) = \frac{\varepsilon}{2} \Delta \rho. \tag{4.6}$$

The previous equation can be identified to the WASEP evolution (2.10) after rescaling the time by a factor $\nu = 1/\varepsilon$. Thus in the hydrodynamic regime, the TASEP can be seen as the limit of the WASEP at large drift. Formally, this means that at the hydrodynamic level the limits $N \to \infty$ and $\nu \to \infty$ can be replaced by $\nu = N \to \infty$.

4.2. Large Deviations

The Jensen-Varadhan functional $\mathcal{J}_{[0,T]}$, introduced in Ref. (24), provides an estimate of the probability that the rescaled particle configuration remains close to a given density profile $\rho(x,t)$ over the macroscopic time interval $[0, T]$

$$\mathbb{Q}^{\varphi}_{N,TN}(\{\eta \sim \rho\}) \approx \exp\left(-N \mathcal{J}_{[0,T]}(\rho)\right), \tag{4.7}$$

with initial data $\varphi(x) = \rho(x, 0)$. The key role of the weak solutions (4.3) in the large deviation regime was understood in Ref. (24): if ρ is not a weak solution then $\mathcal{J}_{[0,T]}(\rho) = \infty$, which means that the probability (4.7) vanishes at an exponential order faster than N. For general weak solutions, the Jensen-Varadhan functional is defined in terms of the Kruzkhov entropy.[36] In this paper, we will simply focus on a few concrete examples for which an explicit representation of $\mathcal{J}_{[0,T]}$ can be stated.

Let ρ be a weak solution made of several shocks and anti-shocks (see (4.4)) and such that ρ remains constant equal to ρ_a and ρ_b in a small neighborhood of the reservoirs during the time interval $[0, T]$ (the latter condition enables us to disregard the boundary effects). If ρ has a single anti-shock of the type (4.4) (i.e. $a > b$) during time T, then the Jensen-Varadhan functional derived in Ref. (24) is given by

$$\mathcal{J}_{[0,T]}(\rho) = TF(a, b)$$

with

$$F(a, b) = g(b) - g(a) - \frac{\sigma(b) - \sigma(a)}{b - a}(h(b) - h(a)), \tag{4.8}$$

where

$$\forall u \in [0, 1], \qquad h(u) = u \log u + (1 - u) \log(1 - u),$$
$$g(u) = u(1 - u) \log \frac{u}{1 - u} - u. \tag{4.9}$$

(Remark that h is the natural entropy associated to the TASEP and that $g' = h'\sigma'$.) If there are several anti-shocks their contributions add up: suppose that the evolution ρ contains ℓ anti-shocks $\{(a_i, b_i)\}_{i\leq\ell}$ each of them maintained during the time interval $[t_i, s_i] \subset [0, T]$, then

$$\mathcal{J}_{[0,T]}(\rho) = \sum_{i=1}^{\ell}(s_i - t_i)F(a_i, b_i). \tag{4.10}$$

In the previous examples, the boundary effects are disregarded. In fact, the Jensen-Varadhan theorem was derived on a torus and does not take into account the influence of the boundaries. Nevertheless (4.7) should also apply in an open system for weak solutions ρ which are smooth at the boundary. Heuristically, the Jensen-Varadhan functional is given by the local cost of each anti-shock thus the regularity of the density at the boundary should be enough to decouple the bulk from the reservoirs. The boundary contribution will be discussed in Section 5.2.

5. FROM WASEP TO TASEP

5.1. Physical Relevance of the Weak Solutions

Some of the weak solutions of the Burgers equation (eg. the anti-shocks) are considered as unphysical and disregarded. On the other hand, we showed in (3.10) that the anti-shocks with discontinuities $(\alpha, 1 - \alpha)$ arise naturally as the limit of the optimal density profiles which realize a current large deviation $\nu\sigma(\alpha)$ when $\nu \to \infty$. Furthermore, the large deviation functional associated to the WASEP converges to the Jensen-Varadhan functional corresponding to the stationary anti-shock with discontinuity $(\alpha, 1 - \alpha)$ (see (3.10) and (4.8)). There are many ways of producing a current deviation even in a stationary regime, but conditionally to observing a current deviation of the form $\nu\sigma(\alpha)$, the typical profiles are close to an anti-shock with discontinuity $(\alpha, 1-\alpha)$: the current deviation selects a specific weak solution.

Our observation shows that the viscosity regularization (4.6) remains valid beyond the law of large number regime and that the Jensen-Varadhan functional can be understood as a limit of the WASEP functional (2.10). Another example of an anti-shock traveling at non-zero velocity will be discussed in the Appendix.

The correspondence between the WASEP and the TASEP process in the large deviation regime has been established at the macroscopic scale, i.e. for the large deviation functionals. At the microscopic level, the correspondence is far from being obvious as the scaling limit for the WASEP (2.8) and TASEP (4.1) are of different nature. We can rewrite (3.10) as

$$\mathbb{P}^{\varphi}_{N,t}\left(\frac{Q_t}{t} \sim \nu q\right) \approx \exp\left(-t\left[\frac{1}{N}G(\nu q) + o\left(\frac{1}{N}\right)\right]\right) \approx \exp\left(-\frac{t\nu}{N}F(\alpha, 1-\alpha)\right). \tag{5.1}$$

The order of the limits is the following: first $t \to \infty$, then $N \to \infty$ and finally $\nu \to \infty$. At least formally, in the limit $\nu = N \to \infty$, the asymptotic (5.1) is consistent with the Jensen-Varadhan theory.

Notice also that for the TASEP, the current distribution in Refs. (15, 10) scales differently depending whether the current is decreased or increased. In (3.18), a similar behavior was pointed out for the WASEP. For ν large, the probability of observing a current deviation νq with $q > 1/4$ is given by

$$\begin{aligned}\mathbb{P}^{\varphi}_{N,t}\left(\frac{Q_t}{t} \sim \nu q\right) &\approx \exp\left(-t\left[\frac{1}{N}G(\nu q) + o\left(\frac{1}{N}\right)\right]\right)\\ &\approx \exp\left(-\frac{t\nu^2}{N}\left(q - \frac{1}{4}\right)^2\right).\end{aligned} \tag{5.2}$$

At least formally, if $\nu = N$ the previous asymptotic is consistent with the scaling expected for the TASEP (see Ref. 15) to entropyincrease the current beyond $1/4$ all the particles have to be accelerated. One should not however expect the leading order in (5.2) with $\nu = N$ to be also valid for the TASEP. Eq. (5.2) implicitly assumes that for large N, the measure becomes locally a product measure. We believe instead that in the TASEP, the optimal measure to increase the current remains different from a product measure even for large N.

5.2. Boundary Contributions

The Jensen-Varadhan functional introduced in Section 4.2 takes into account only the bulk large deviations and not the boundary contribution. At present, there is no microscopic derivation of the large deviations for the TASEP in contact with reservoirs. For diffusive models, the large deviation functional of the system with reservoirs is simply the functional in the bulk with the constraint that the densities at the reservoirs are fixed equal to ρ_a and ρ_b (see (2.14) or Ref. (5) for a rigorous derivation in the case of the SSEP). The Jensen-Varadhan functional cannot be generalized in this way, as we know that even in the stationary regime, the reservoirs do not impose a fixed density: the steady state of the TASEP can be discontinuous at the boundary.[14,29,31,34,35]

In Section 3.3, we computed the probability of observing a sharp variation of the density at the boundary for large ν. As the Jensen-Varadhan functional can be interpreted as the limit of the WASEP large deviation functional in the bulk, we conjecture that (3.21), (3.24–3.26) provide also the right expression for the boundary contribution. If this is the case, the functional associated to the system with reservoirs should be the sum of the Jensen-Varadhan functional (in the bulk) and of the boundary effects.

The previous conjecture cannot be validated without a proper derivation from the microscopic model. Nevertheless, further comments can be made to justify our guess. We will focus on the case $\rho_a > 1/2$. For the TASEP on the half-line $\{1, 2, \ldots\}$, one can find, for any density γ in $[\frac{1}{2}, 1]$, an invariant measure with mean asymptotically close to γ.[31] This implies that there is no cost to maintain a discontinuity (ρ_a, γ) for any γ in $[\frac{1}{2}, 1]$ as predicted by (3.21). For $\gamma \in [1 - \rho_a, \frac{1}{2}]$, (3.21) asserts that $1 - \gamma$ plays the role of an effective boundary. The boundary layer is made of two pieces, first a jump from ρ_a to $1 - \gamma$ (which costs nothing as $1 - \gamma > 1/2$) and then an anti-shock with zero velocity between $1 - \gamma$ and γ which has a cost given by the Jensen-Varadhan functional. A similar interpretation holds for the expressions (3.24–3.26).

6. CONCLUSION

In the present paper, we have analyzed the large deviation functional of the current for the WASEP in contact with reservoirs. In the large drift limit, we have obtained an expression (3.10) similar to the Jensen-Varadhan functional (4.8). We have also obtained the cost (3.21), (3.24–3.26) of maintaining boundary layers in the large drift limit and we believe that these expressions should remain valid for the TASEP. Of course, it would be necessary to validate our expressions from a calculation starting from the microscopic model. For general weak solutions, the Jensen-Varadhan functional has to be formulated in terms of the Kruzkhov entropy in Ref. (24) It would be interesting to relate the boundary costs (3.21), (3.24–3.26) to the more general entropy condition introduced for open systems in Refs. (2, 37).

A strong connection was derived in Refs. (4, 23) between the hydrodynamic large deviation functional and the one associated to the stationary measure: a density deviation in the stationary measure can be interpreted as the optimal space/time cost to produce this deviation starting from the steady state. Thus, one could try to see whether the optimal trajectory to generate a given steady state fluctuation in the WASEP converges to the optimal trajectory in the TASEP (after all, if the large ν limit of the WASEP describes the TASEP steady state at the level of the density large deviation functional,[20] one might expect the large drift limit to hold at the level of the whole trajectories).

Our calculation of the large deviations of the integrated current relies on the hypothesis that the optimal density profiles are time-independent and that the measure conditioned on a given current is in local equilibrium. This led in particular to (5.2) which we do not expect to hold in the TASEP: as for the ring geometry, the measure conditioned on such an increase of the current should be significantly different from a Bernoulli measure.

Even if we think that the current large deviations in the TASEP are correctly given by (3.9), (3.10), (3.16) (i.e. the limit $\nu \to \infty$ of the WASEP), we believe that the distribution of the fluctuations of the TASEP are not of diffusive nature and cannot be understood by taking simply the large ν limit (for example in the WASEP the fluctuations of density are Gaussian[12] whereas in the TASEP they are not in the case of open boundary conditions.[13]

Finally, it would be interesting to extend our approach to particle systems with more general jump rates. Hydrodynamic equations were derived for a broad class of microscopic models.[21,26,28] In this case, there exists a density dependent diffusion coefficient $D(\rho)$ and a conductivity coefficient $\sigma(\rho)$ such that (2.10) becomes

$$\forall (x,t) \in \mathbb{D} \times [0,T], \qquad \partial_t \rho = \nabla_x (D(\rho)\nabla_x \rho) - \nu \nabla_x (\sigma(\rho)). \tag{6.1}$$

The counterpart of the large deviation functional (2.14) can be expressed in terms of the coefficients D and σ[6,7,8,9]

$$\mathcal{I}^{\nu}_{[0,T]}(j,\rho) = \int_0^T dt \int_0^1 \mathrm{d}x \, \frac{(j(x,t) - \nu\sigma(\rho(x,t)) + D(\rho(x,t))\nabla_x \rho(x,t))^2}{2\,\sigma(\rho(x,t))}. \tag{6.2}$$

One can address the same type of questions as in this paper and minimize this functional under a current deviation constraint. The limiting expression (obtained for large drift ν) might provide the extension of the Jensen-Varadhan functional to more general asymmetric dynamics.

7. APPENDIX

In Section 3.2, a constant flux was imposed and we proved the convergence of the large deviation functionals for ν large. We are now going to investigate the effect of a time dependent current constraint. More precisely, we are looking for the optimal way of producing a current with different values on both sides of a singularity evolving at velocity v

$$j(x,t) \sim \begin{cases} \nu\sigma(a), & \text{if } x \ll vt, \\ \nu\sigma(b), & \text{if } x \gg vt. \end{cases} \tag{7.1}$$

For simplicity, we consider the hydrodynamic evolution in $\mathbb{R}$ instead of the finite system $\mathbb{D} = [0, 1]$. The large deviation functional introduced in (2.14) can be defined in $\mathbb{R}$. We would like to minimize the functional $\mathcal{I}^{\nu}_{[0,T]}$ under the current constraint (7.1) and find the optimal profile ρ and the flux j (both depending on ν) for ν large. By construction one has

$$\partial_t \rho = -\nabla j.$$

Furthermore, we restrict the class of density profiles to the traveling waves so that

$$\rho(x,t) = \rho(x - \nu v t), \quad \text{and} \quad \nu v \rho' = j'.$$

Integrating the previous equation, we deduce that $j = \nu(v\rho + c)$. As the current is imposed at infinity by the constraint (7.1), the parameters v, c are given by

$$v = \frac{\sigma(a) - \sigma(b)}{a - b} \quad \text{and} \quad c = \frac{a\sigma(b) - b\sigma(a)}{a - b} = -va + \sigma(a) = -vb + \sigma(b).$$

We remark that this imposes the same condition on the velocity as in (4.5).

The large deviation functional minimized over the traveling waves is given by

$$\begin{aligned} G(\nu, a, b) = & \lim_{T\to\infty} \inf_{\rho} \left\{ \frac{1}{T} \int_0^T \mathrm{d}t \int_{\mathbb{R}} \mathrm{d}x \, \frac{\left(j - \nu\sigma(\rho) + \frac{1}{2}\rho'\right)^2}{2\,\sigma(\rho)} \right\} \\ = & \inf_{\rho} \left\{ \int_{\mathbb{R}} \mathrm{d}x \, \frac{\left(\nu(c + v\rho) - \nu\sigma(\rho) + \frac{1}{2}\rho'\right)^2}{2\,\sigma(\rho)} \right\} \end{aligned} \tag{7.2}$$

where we used the explicit form of j in terms of ρ.

We are going to recover the convergence to the Jensen-Varadhan functional (4.8). The optimal cost is given by the profile ρ which minimizes (7.2). We decompose G into two terms

$$\begin{aligned} G_1 &= \frac{1}{2} \int_{\mathbb{R}} \mathrm{d}x \, \frac{\nu^2 (c + v\rho - \sigma(\rho))^2}{\sigma(\rho)} + \frac{(\rho')^2}{4\sigma(\rho)} \quad \text{and} \\ G_2 &= \nu \int_{\mathbb{R}} \mathrm{d}x \, \frac{(c + v\rho - \sigma(\rho))\rho'}{2\sigma(\rho)}. \end{aligned} \tag{7.3}$$

The functional G_2 does not depend on the profile ρ

$$G_2 = \frac{\nu}{2}\int_a^b \mathrm{d}\rho\,\left(\frac{c}{\sigma(\rho)} - 1 + v\frac{\rho}{\sigma(\rho)}\right) = \frac{\nu}{2}\left[c\log\frac{\rho}{1-\rho} - \rho - v\log(1-\rho)\right]_a^b.$$

Using the explicit formula for c, we get

$$G_2 = \frac{\nu}{2}\left[g(b) - g(a) - v(h(b) - h(a))\right] = \frac{\nu}{2}F(a,b).$$

It remains to determine the optimal profile ρ by minimizing G_1. Following the same computation as in (3.3), we see that there is a constant K such that

$$\left(\frac{1}{2}\rho'\right)^2 = \nu^2(c + v\rho - \sigma(\rho))^2 + 2K\sigma(\rho).$$

The constraint (7.1) at infinity implies that $K = 0$. Thus we get

$$G_1 = \frac{\nu}{2}\int_{\mathbb{R}} \mathrm{d}x\,\frac{(c + v\rho - \sigma(\rho))\rho'}{\sigma(\rho)} = G_2.$$

Combining the previous results, we get

$$G(\nu q) = \begin{cases} \nu F(a,b), & \text{if } a > b \\ 0, & \text{if } a < b \end{cases}.$$

For ν large (and in fact for all ν), $G(\nu q)/\nu$ is equal to the Jensen-Varadhan functional (4.7), (4.8). As expected the shocks ($a < b$) behave differently from the anti-shocks ($a > b$). The optimal profile has a sharp slope in an interval of size $1/\nu$ centered around vt and the cost to maintain the anti-shock is essentially located in this region. This confirms the Jensen-Varadhan theory which asserts that the contribution of the anti-shocks decouple (4.10). For this reason the previous approximation procedure should be also valid for a more general deviation of the current with several discontinuities: in the limit $\nu \to \infty$, each anti-shock should contribute independently to the large deviation functional.

In Refs. (9), it was argued that some current deviations for the WASEP on a ring are achieved by a traveling wave. In the large drift limit, the exponential cost of these traveling waves was also related to the Jensen-Varadhan functional.

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